Quaestiones Mathematicae

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tqma20

On higher order generalized geometric polynomials with shifted parameters

José A. Adell, Beáta Bényi & Sithembele Nkonkobe

To cite this article: José A. Adell, Beáta Bényi & Sithembele Nkonkobe (2023) On higher order generalized geometric polynomials with shifted parameters, Quaestiones Mathematicae, 46:3, 551-567, DOI: 10.2989/16073606.2022.2035843

To link to this article: https://doi.org/10.2989/16073606.2022.2035843

Published online: 14 Feb 2022.

Quaestiones Mathematicae 2023, 46(3): 551–567.

https://doi.org/10.2989/16073606.2022.2035843

ON HIGHER ORDER GENERALIZED GEOMETRIC POLYNOMIALS WITH SHIFTED PARAMETERS

José A. Adell

Departamento de Métodos Estadísticos, Facultad de Ciencias, Universidad Zaragoza, Spain.
E-Mail adell@unizar.es

Beáta Bényi

Department of Hydraulic Engineering, University of Public Service, Baja, Hungary.
E-Mail benyi.beata@uni-nke.hu

Sithembele Nkonkobe*

Department of Mathematical Sciences, Sol Plaatje University, Kimberley, South Africa.
E-Mail snkonkobe@gmail.com

Abstract. We study a special class of higher order generalized geometric polynomials. Based on our combinatorial interpretation of labeled barred preferential arrangements, we prove several recursions. We also study the polynomials from a probabilistic point of view, and show how our polynomials can be written in terms of the expectation of a random descending factorial involving the negative binomial process. Using techniques of probability theory, we derive identities, in particular we extend Nelsen's theorem.

Mathematics Subject Classification (2020): 05A15, 05A19, 60C99.

Key words: Geometric polynomials, barred preferential arrangements, negative binomial process.

1. Introduction. Geometric polynomials are defined as (see [5, 17, 31]),

$$w_n(x) = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\} x^k, \tag{1}$$

where $\left\{ {n \atop k} \right\}$ denotes the Stirling number of the second kind, which counts the number of ways the set $\{1, 2, \ldots, n\}$ can be partitioned into k non-empty blocks. Geometric polynomials can be seen as the generalization of the generating function $(2 - e^x)^{-1}$ (see for instance, [5, 6, 14, 20]), since the exponential generating function of the

*Corresponding author.

geometric polynomials (1) is given by (c.f.[5]):

$$\sum_{n=0}^{\infty} w_n(x)\frac{t^n}{n!} = \frac{1}{1-x(e^t-1)}. \tag{2}$$

Geometric polynomials are related to the geometric series by (c.f.[5])

$$(xD)^m\left\{\frac{1}{1-x}\right\} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x} w_n\left(\frac{x}{1-x}\right), \quad |x|<1.$$

They seem to appear first in Euler's work [15, p. 389]. In the literature, the polynomials (1) are also known as *Fubini polynomials*, since the values for $x = 1$, $w_n(1)$, are called the *Fubini numbers* by Comtet: $1, 1, 3, 13, 75, 541, 4683, \ldots$ [A000670] in [32].

Fubini numbers enumerate, among other combinatorial objects, ordered partitions, that is, partitions in that the order of blocks is also taken into account. For this reason, $w_n(1)$ are also known as *ordered Bell numbers*, *preferential arrangement numbers*, or *nth geometric numbers* in the literature (see [16, 17, 29, 31]).

The n^{th} geometric numbers, $w_n(1)$, appear also in the evaluation of the series

$$\frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k} = w_n(1),$$

which are n^{th} moments of the random variable having the geometric distribution with success probability of 1/2 (c.f.[12]).

The polynomials (2) and their different generalizations have been extensively studied in the literature (see, [5, 6, 9, 14, 20, 21, 22, 23, 25]).

For instance, higher order geometric polynomials

$$w_n^{(r)}(x) = \sum_{k=0}^{n} \left\{ n \atop k \right\} r^{(k)} x^k, \quad r>0,$$

where $r^{(k)} = r(r+1)(r+2)\cdots(r+k-1)$, were introduced in [5], and also studied in [22]. The numbers $w_n^{(r)}(1)$ count barred preferential arrangements, as shown in [2].

Some generalizations of $w_n(x)$ are based on the generalization of the Stirling number of the second kind $\left\{ n \atop k \right\}$. Hsu and Shiue in [19] showed that various generalizations of the Stirling numbers (both first and second kind) can be unified by introducing a variant involving three parameters, α, β, γ, as follows:

$$(t|\alpha)_n = \sum_{k=0}^{n} S(n,k,\alpha,\beta,\gamma)(t-\gamma|\beta)_n,$$

where $(t|\alpha)_n$ is the generalized factorial polynomial $(t|\alpha)_n = \prod_{k=0}^{n-1}(t-k\alpha)$, $n \geq 1$, and α, β, γ are real or complex numbers not all equal to zero. A combinatorial and statistical approach to this unified generalization was given by Corcino et al. in [11]. The generalized Stirling numbers play, for instance, a key role in one of the generalizations of the Mellin derivative (see [23]),

$$\left(\beta x^{1-\alpha/\beta} D\right)^n \left[x^{\gamma/\beta} f(x)\right] = x^{(\gamma-n\alpha)/\beta} \sum_{k=0}^{n} S(n,k;\alpha,\beta,\gamma)\beta^k x^k f^{(k)}(x). \tag{3}$$

Using the generalized Mellin derivative (3), Kargin and Cekim in [21] introduced *higher order generalized geometric polynomials*, $w_n^{(\lambda)}(x;\alpha,\beta,\gamma)$, defined by

$$\sum_{n=0}^{\infty} w_n^{(\lambda)}(x;\alpha,\beta,\gamma)\frac{t^n}{n!} = \frac{(1+\alpha t)^{\gamma/\alpha}}{(1-x((1+\alpha t)^{\beta/\alpha}-1))^{\lambda}}. \tag{4}$$

Based on the properties of this new family of polynomials, they presented several interesting applications.

Higher order generalized geometric polynomials, $w_n^{(\lambda)}(x;\alpha,\beta,\gamma)$, were investigated from a combinatorial point of view in [28] and also studied in [21].

Motivated by the work of Mihoubu and Taharbouchet in [25], we study higher order generalized geometric polynomials, where the parameters λ and γ are shifted. More precisely, $w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j)$, which we call *higher order generalized geometric polynomials with shifted parameters.*

The generating function is,

$$\sum_{n=0}^{\infty} w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j)\frac{t^n}{n!} = \frac{(1+\alpha t)^{(j+\gamma)/\alpha}}{(1-x((1+\alpha t)^{\beta/\alpha}-1))^{\lambda+r}}, \tag{5}$$

where r is a nonnegative integer, and $\alpha,\beta,\gamma,\lambda$ are positive integers, such that $\alpha|\beta$ and $\alpha|\gamma$. Also, in (5) $j = r\beta - r\alpha$.

In the second section, we present the combinatorial interpretation of the higher order generalized geometric polynomials with shifted parameters. Based on that, we derive properties, recursions, and formulas. In the last section we consider our polynomials from a probabilistic point of view. In particular, we show that such polynomials can be written in terms of the expectation of a random descending factorial involving the negative binomial process. Moreover, the probabilistic representation allows us to show various identities and to extend Nelsen's theorem for arbitrary polynomials in a simple way.

2. Combinatorial results

The main goal of this section is to give a combinatorial interpretation of our polynomials, $w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j)$, and to derive some recursions. To this end, we need to recall some known facts.

The objects we will use are the so called *barred preferential arrangements. Preferential arrangements* are ordered partitions, e.i., partitions such that the order of the blocks counts. The use of the different names (ordered partitions and preferential arrangements) emphasizes different points of view. The ways how competitors can be ranked in a competition allowing ties is a preferential arrangement. One thinks on an ordered partition as creating blocks, and then permuting the blocks, while on a preferential arrangement as choosing elements for the first rank, then for the second rank and so on (see [17, 24]).

We obtain a *barred preferential arrangement* (BPA) by inserting bars in between (before or after) the blocks of a preferential arrangement. Barred preferential arrangements seem to appear first in [2, 31] and are studied in [8, 28, 29].

In this paper, we are dealing with *labeled barred preferential arrangements.*

Definition 1. A *labeled barred preferential arrangement* is a barred preferential arrangement with a labeling on the bars, i.e., an ordered partition with labeled bars inserted between (before or after) the blocks.

Example 2. For $n = 6$ a labeled barred preferential arrangement (LBPA) is given for instance: $B_1 = \{3, 5\}\ \ \{2\}|^1\ \{1\}\ \ |^2\{4, 6\}$, or $B_2 = |^3\{1, 3, 6\}\ \ \{4\}\ \ \{2\}|^1\ \ |^2\{5\}$.

We emphasize that we can view these objects from different points of view. For instance, B_2 is obtained by inserting the labeled bars $|^1$, $|^2$ and $|^3$ in between the blocks of the ordered partition: $\{1, 3, 6\}\{4\}\{2\}\{5\}$, or focusing on the bars, we can say we inserted the subsets of $[n]$ in between the arrangement of labeled bars, $|^3\ \ |^1\ \ |^2$. We see that λ bars separate an LBPA into $\lambda + 1$ pieces. We call these pieces *sections.* A section may be empty as the section between the bars $|^1\ \ |^2$ in B_2 or it contains subsets of $[n]$ in a certain order. We will use both views in our arguments. Labeled barred preferential arrangements were also used in [4] for the study of symmetrized poly-Bernoulli numbers.

The other ingredient that is important for understanding our combinatorial model is the unified generalization of Stirling numbers with three parameters $S(n, k, \alpha, \beta, \gamma)$ introduced by Hsu and Shiue in [19].

The pair $S(n, k, \alpha, \beta, \gamma)$ is a formal generalization of the two well known algebraic identities involving Stirling numbers:

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} (x)_k \quad \text{and}$$
$$x^{(n)} = \sum_{k=0}^{n} \left[{n \atop k} \right] x^k,$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ is the falling factorial and $x^{(n)} = x(x+1)\cdots(x+n-1)$ is the rising factorial.

The generalized factorial of t with increment α for an integer $n \geq 1$ is defined as

$$(t|\alpha)_n = t(t-\alpha)\cdots(t-n\alpha+\alpha),$$

and $(t|\alpha)_0 = 1$. The polynomials that arise in the basic relations between Stirling numbers are all special cases of $(t|\alpha)_n$ with special values of α: $t^n = (t|0)_n$, $(t)_n = (t|1)_n$ and $t^{(n)} = (t|-\alpha)_n$.

The generalized Stirling pair $\{S^1, S^2\} = \{S(n,k;\alpha,\beta,\gamma), S(n,k;\beta,\alpha,-\gamma)\}$ with three parameters are defined in [19] by

$$(t|\alpha)_n = \sum_{k=0}^{n} S^1(n,k)(t-\gamma|\beta)_k \quad \text{and}$$
$$(t|\beta)_n = \sum_{k=0}^{n} S^2(n,k)(t+\gamma|\alpha)_k,$$

where $n \geq 1$ is an integer and α, β, γ are any real or complex numbers with $(\alpha,\beta,\gamma) \neq (0,0,0)$.

Corcino et al. [11] have introduced a combinatorial model for studying generalized Stirling numbers. They showed that $m!\beta^m S(n,m;\alpha,\beta,\gamma)$ is the number of ways to distribute n distinct balls, one ball at a time into $k+1$ distinct cells, where the first k of them have β distinct compartments and a last cell with γ distinct compartments such that

(1) the compartments in each cell are given cyclic ordered numbering,

(2) the capacity of each compartment is limited to one ball,

(3) each successive α available compartment in a cell can only have the leading compartment getting a ball,

(4) the first m cells are non-empty.

For instance, suppose the first ball lands in the 4th compartment of the 3th cell. The next α compartments, i.e., the compartments numbered 5, 6, …, $\alpha+3$ will be closed. Suppose the second ball lands in compartment $\beta-2$ in the 3th cell. Then the compartments $\beta-1$, β, 1, 2, 3, $\alpha+4$, …, $2\alpha-3$ will be closed and so on. We recall the generating function of these numbers.

Lemma 3. ([19]) *For real or complex α, β, γ, we have*

$$(1+\alpha t)^{\frac{\gamma}{\alpha}} \left[\frac{(1+\alpha t)^{\frac{\beta}{\alpha}}-1}{\beta}\right]^m = m! \sum_{n=0}^{\infty} S(n,m,\alpha,\beta,\gamma)\frac{t^n}{n!}. \tag{6}$$

The number of distributions satisfying properties (1), (2) and (3), but not requiring (4), i.e. the first m cells may be empty, is given by $(\beta m+\gamma|\alpha)_n$ (c.f.[11]).

We consider the r-version of this model, i.e., we require that the elements $\{1,2,\ldots,r\}$ be in distinct cells in analogue to the combinatorial interpretation of the r-Stirling numbers introduced by Broder [7]. We call our model the $r-(\alpha,\beta,\gamma)$-*model.*

Definition 4. ($r-(\alpha,\beta,\gamma)$-model) We distribute $n+r$ balls into $m+r+1$ cells, where each cell is built up with cyclically labeled compartments. The first (left to right) $m+r$ cells have β such compartments and the last cell has γ compartments (will also be referred to as the γ-cell) such that:

(1) the capacity of each compartment is limited to one ball,

(2) on each available consecutive α compartments only the first compartment may be occupied,

(3) the first $m+r$ cells are non-empty,

(4) the first r balls are on distinct cells among the first $m+r$ cells.

We call the elements $\{1,2,\ldots,r\}$ *special elements* and the cells containing a special element *special cells*. The elements $\{r+1,\ldots,n+r\}$ are called *non-special elements* and the cells not containing any of the special elements, the *non-special cells*.

Let $j=r(\beta-\alpha)$ and $(m+r)!\beta^m S(n,m,\alpha,\beta,\gamma+j)$ denote the number of ways n balls can be distributed in a $r-(\alpha,\beta,\gamma)$-model.

The next theorem shows the relation with the Corcino-Hsu-Tan model.

Theorem 5. *For $n\geq r$, we have*

$$\beta^m S(n,m,\alpha,\beta,\gamma+j)=\sum_{k=0}^{n}\binom{n}{k}\beta^m S(k,m,\alpha,\beta,\gamma)(j|\alpha)_{n-k}. \tag{7}$$

Proof. We consider a partition of $[n+r]=\{1,2,\ldots,n+r\}$ into $m+r+1$ cells, where $m+r$ of them have β compartments and one has γ compartments (γ-cell).

Let k be the number of elements that are contained in the non-special cells. The number of ways of distributing these k elements in the non-special cells is given by $\beta^m S(k,m,\alpha,\beta,\gamma)$. Put into each special cell one of the r special elements and distribute the remaining $n-k$ non-special elements into one of these cells. This can be done in $(r(\beta-\alpha)|\alpha)_{n-k}$ ways (since now empty cells are also allowed). □

We now give a closed formula for the higher order generalized geometric polynomials with shifted parameters $w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j)$.

Theorem 6. *For $n\geq 0$, we have*

$$w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j)=\sum_{m=0}^{n}(m+r+\lambda-1)!\beta^m S(n,m,\alpha,\beta,\gamma+j)x^m. \tag{8}$$

Proof. By Lemma 3, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{m+r+\lambda-1}{m+r}(m+r)!\beta^m S(n,m,\alpha,\beta,\gamma+j)x^m\frac{t^n}{n!}=\\ &\sum_{m=0}^{\infty}\binom{m+r+\lambda-1}{m+r}(m+r)!\frac{(1+\alpha t)^{\frac{j+\gamma}{\alpha}}}{m!}[((1+\alpha t)^{\frac{\beta}{\alpha}}-1)x]^m.\end{aligned} \tag{9}$$

Thus,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{m+r+\lambda-1}{m+r}(m+r)!(\lambda-1)!\beta^m S(n,m,\alpha,\beta,\gamma+j)x^m\frac{t^n}{n!}= \tag{10}$$
$$\frac{(1+\alpha t)^{\frac{j+\gamma}{\alpha}}}{[1-((1+\alpha t)^{\frac{\beta}{\alpha}}-1)x]^{\lambda+r}}. \qquad \square$$

When x is a nonnegative integer we can interpret the numbers $w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j)$ combinatorially. Let $\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma,\lambda;x)$ denote the set of labeled barred preferential arrangements where the underlying partition satisfies the conditions of the $r-(\alpha,\beta,\gamma)$ model and the non-special cells are colored with a color out of x available colors.

Lemma 7. *We have*

$$|\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma,\lambda;x)| = w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j).$$

Proof. To see this, recall that $\beta^m S(n,m,\alpha,\beta,\gamma+j)$ is the number of ways of partitioning an $(n+r)$ element set into $m+r+1$ unordered cells satisfying the conditions of the $r-(\alpha,\beta,\gamma)$ model. We obtain from such a partition an LBPA if we permute the $m+r$ cells together with the $\lambda-1$ labeled bars. $\square$

We present some recursive relations and provide for all a combinatorial proof using the above interpretation.

Theorem 8. *For $n\geq 0$, we have*

$$w_{n+1}^{(\lambda+r)}(x;\alpha,\beta,\gamma+j) = \tag{11}$$
$$(j+\gamma)w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j-\alpha)+x\beta(\lambda+r)w_n^{(\lambda+1+r)}(x;\alpha,\beta,\gamma+j+\beta-\alpha).$$

Proof. We enumerate the set $\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma,\lambda;x)$ based on the position of the $(n+1)$th non-special element.

Case 1: The $(n+1)$th element is in one of the special cells. Given an LBPA from the set $\mathcal{H}_n^{r,j-\alpha}(\alpha,\beta,\gamma,\lambda;x)$ one can insert the $(n+1)th$ element into a special cell in j ways.

Case 2: The $(n+1)$th element is in the γ-cell. Given an LBPA from the set $\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma-\alpha,\lambda;x)$, $(n+1)$ can be inserted into a γ cell in γ ways, choosing the compartment for it.

Case 3: The $(n+1)$th element is in one of the non-special cells having β compartments. Call the cell B. Inside the cell B the $(n+1)$th element can be placed in β ways. The cell B can be colored in x ways. Think of the γ-cell and the cell B as a single unit, call it A. Note that A has $\gamma+\beta-\alpha$ available compartments after the $(n+1)$th element has been placed. On both sides of B, cells can be formed, hence, B acts as an extra bar. Thus, the number of LBPAs formed in this case is $x\beta|\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma+\beta-\alpha,\lambda+1;x)|$. $\square$

The next theorem gives a similar recursion. However some additional ideas are involved.

Theorem 9. *For $n \geq 0$, we have*

$$w_{n+1}^{(\lambda+r)}(x;\alpha,\beta,\gamma+j) = \gamma\sum_{k=0}^{n}\binom{n}{k}(\gamma-\alpha|\alpha)_k w_{n-k}^{(\lambda+r)}(x;\alpha,\beta,j) + jw_n^{(\lambda+r)}(x;\alpha,\beta,\gamma-\alpha+j) + \\ + x\beta\sum_{k=0}^{n}\binom{n}{k}(\beta-\alpha|\alpha)_k(\lambda+r)w_{n-k}^{(\lambda+1+r)}(x;\alpha,\beta,\gamma+j).$$

Proof. We count the elements in the set $\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma,\lambda;x)$ also according the position of the $(n+1)$th non-special element.

Case 1: The $(n+1)$th element is in the cell having γ compartments. There are γ ways of placing the $(n+1)$th element into the γ-cell. Let k be the number of other non-special elements in the γ-cell. These k elements can be chosen in $\binom{n}{k}$ ways. The number of ways of placing the k elements is $(\gamma-\alpha|\alpha)_k$. The remaining $n-k$ elements form a LBPA with an empty γ-cell, an element from the set $\mathcal{H}_{n-k}^{r,j}(\alpha,\beta,0,\lambda;x)$. Thus, the number of ways of forming LBPAs in this case is

$$\gamma\sum_{k=0}^{n}\binom{n}{k}(\gamma-\alpha|\alpha)_k w_{n-k}^{(\lambda+r)}(x;\alpha,\beta,j).$$

Case 2: The $(n+1)$th element is on one of the r special cells. Given an element from the set $\mathcal{H}_n^{r,j-\alpha}(\alpha,\beta,\gamma,\lambda;x)$, the $(n+1)$th element can be placed into the special cells in α ways. Thus, LBPAs can be formed in this case in

$$jw_n^{(\lambda+r)}(x;\alpha,\beta,\gamma-\alpha+j)$$

ways.

Case 3: The $(n+1)$th element is in one of the non-special cells having β compartments. The compartment and color of the cell to which the $(n+1)$th element belongs to can be chosen in $x\beta$ ways, call this cell B. From the n other non-special elements choose k to be in the same cell as the $(n+1)$th element. The k elements can be chosen in $\binom{n}{k}$ ways. The k elements can be placed into B in $(\beta-\alpha|\alpha)_k$ ways. The remaining $n-k$ elements form an LBPA of the set $\mathcal{H}_{n-k}^{r,j}(\alpha,\beta,\gamma,\lambda+1;x)$ To the right and to the left of B cells can be formed. Hence, B acts as an extra bar. Thus, in this case the number is

$$x\beta\sum_{k=0}^{n}\binom{n}{k}(\beta-\alpha|\alpha)_k(\lambda+r)w_{n-k}^{(\lambda+1+r)}(x;\alpha,\beta,\gamma+j). \qquad \square$$

Next, we give a formula for the case when $\gamma=0$, i.e., the model does not contain an extra (possible empty) cell with γ compartments.

THEOREM 10. *For $n \geq 0$, we have*

$$w_n^{(\lambda+r)}(x;\alpha,\beta,j) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (\gamma|\alpha)_{n-k} w_k^{(\lambda+r)}(x;\alpha,\beta,\gamma+j). \tag{12}$$

Proof. Let $\mathcal{B}_k$ be the number of barred preferential arrangement of the set $\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma,\lambda;x)$ having $(n-k)$ elements in their γ-cells. So, $|\mathcal{B}_k| = \binom{n}{n-k}|\mathcal{H}_k^{r,j}(\alpha,\beta,\gamma,\lambda;x)|(\gamma|\alpha)_{n-k}$. The application of the inclusion-exclusion principle completes the proof. □

Our next recursion is based on the order of the objects in the LBPA model.

THEOREM 11. *For $n \geq 0$, we have*

$$w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j) = w_n^{(\lambda-1+r)}(x;\alpha,\beta,\gamma+j) + x\sum_{k=1}^{n} \binom{n}{k} (\beta|\alpha)_k w_{n-k}^{(\lambda+r)}(x;\alpha,\beta,\gamma+j). \tag{13}$$

Proof. Recall that an element of the set $\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma,\lambda;x)$ is an arrangement of three kinds of objects, special cells, non-special cells and labeled bars. Hence, the set $\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma,\lambda;x)$ can be partitioned into three classes based on the first object from left to right in an element of the set $\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma,\lambda;x)$. Namely, a labeled bar, a special cell, or a non-special cell.

Case 1. The first object is a labeled bar. There are $\lambda - 1$ ways of labelling the bar. Hence, the number of barred preferential arrangements with a bar on the far left is

$$(\lambda-1)\sum_{m=0}^{n}(m+r+\lambda-2)!\beta^m S(n,m,\alpha,\beta,\gamma+j)x^m.$$

Case 2. The first object is a special cell. There are r ways of choosing a special cell. The number of barred preferential arrangements with a special cell as a first object is

$$r\sum_{m=0}^{n}(m+r+\lambda-2)!\beta^m S(n,m,\alpha,\beta,\gamma+j)x^m.$$

Case 3. The first object is a non-special cell. We denote this first non-special cell from left to right by B. The cell B can be colored in x ways. Say there are k elements in B. Surely the non-special cell is nonempty, hence k runs from 1 to n. The k elements can be chosen in $\binom{n}{k}$ ways. The k elements going into B can be arranged in $(\beta|\alpha)_k$ ways. Barred preferential arrangements can be formed in

$$x\sum_{k=1}^{n}\binom{n}{k}(\beta|\alpha)_k|\mathcal{H}_{n-k}^{r,j}(\alpha,\beta,\gamma,\lambda;x)|$$

in this class. □

Special cases of the recursion given in Theorem 11 can be found in the literature. For instance, Theorem 3.3 of [10], Proposition 4 of [14], Equation 4.9 of [13], Equation 6 of [33], and Theorem 3.2 of [28].

Finally, we want to show how our model can give a combinatorial explanation of a generalization of Nelsen's theorem. In [18], Nelsen conjectured that

$$\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}(-1)^{k-s}(\gamma+s)^n=\frac{1}{2}\sum_{s=0}^{\infty}\frac{(\gamma+s)^n}{2^s}, \tag{14}$$

for $\gamma \in \mathbb{R}$, and non-negative integer n.

In [26], Donald Knuth et al. give several alternative proofs of the conjecture, of which none includes a combinatorial interpretation. In order to reveal the connection, we derive an appropriate formula and prove it combinatorially.

PROPOSITION 12. *For $n \geq 0$, we have*

$$\begin{aligned}(15)\quad & w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j)=\\ &\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}x^k(-1)^{k-s}\sum_{d=0}^{n}\binom{n}{d}(s\beta|\alpha)_{n-d}w_d^{(\lambda-1+r)}(x;\alpha,\beta,\gamma+j).\end{aligned}$$

Proof. First we count the number of LBPAs such that the first objects are non-special cells from which may be some of them empty and then we apply the inclusion-exclusion principle to obtain the result. We choose d elements out of the n non-special elements in $\binom{n}{d}$ ways and form a LBPA of the $d+r$ elements such that the first object is a bar or a special cell. Let $\mathcal{R}_d(\alpha,\beta,\gamma)$ denote the set of such LBPAs. According to the argument of Theorem 11, we have

$$|\mathcal{R}_d(\alpha,\beta,\gamma)|=(\lambda+r-1)\sum_{m=0}^{n}(m+r+\lambda-2)!\beta^m S(d,m,\alpha,\beta,\gamma+j)x^m.$$

To the left of the LBPA of $\mathcal{R}_d(\alpha,\beta,\gamma)$ we insert k non-special cells formed from the remaining $n-d$ elements such that s cells are non-empty, which can be done in $(s\beta|\alpha)_{n-d}$ ways. Finally, we apply the inclusion-exclusion principle to obtain the number of LBPAs where there are no empty non-special cells to the left of the LBPA from $\mathcal{R}_d(\alpha,\beta,\gamma)$ with that we started the construction:

$$|\mathcal{H}_n^{r,j}(\alpha,\beta,\gamma,\lambda;x)|=\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}x^k(-1)^{k-s}\sum_{d=0}^{n}\binom{n}{d}(s\beta|\alpha)_{n-d}|\mathcal{R}_d(\alpha,\beta,\gamma)|.$$ □

From the generating function (10) we obtain:

$$w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j)=\sum_{k=0}^{\infty}\frac{x^k}{(1+x)^{k+1}}w_n^{(\lambda+r-1)}(x;\alpha,\beta,\gamma+j+k\beta). \tag{16}$$

Some special cases of (16) appear in the literature. For instance, Equation 10 of [3], Theorem 3.2 of [10], Proposition 4.12 of [13], and Corollary 2 of [25].

By (16) and Proposition 12, we have

$$\sum_{k=0}^{n}\sum_{s=0}^{k}\binom{k}{s}x^k(-1)^{k-s}\sum_{d=0}^{n}\binom{n}{d}w_d^{(\lambda+r-1)}(x;\alpha,\beta,\gamma+j)(s\beta|\alpha)_{n-d}=\sum_{k=0}^{\infty}\frac{x^k}{(1+x)^{k+1}}w_n^{(\lambda+r-1)}(x;\alpha,\beta,\gamma+j+k\beta). \tag{17}$$

Remark that (17) is a generalization of Nelsen's identity in (14). An analogue of the special case $\alpha=0$, $\beta=1$, $x=1$ $r=1$ of (17) appears in Theorem 4.1 of [27, p. 48], another in [30] for $\alpha=0, x=1$. In both discussions the identities are connected to the number of barred preferential arrangements (unlabeled version).

3. Probabilistic results In this section, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For any $m\in\mathbb{N}$, let $(Z_m(x))_{x\geq 0}$ be the negative binomial process defined as

$$P(Z_m(x)=i)=\binom{-m}{i}\left(-\frac{x}{x+1}\right)^i\left(\frac{1}{x+1}\right)^m,\quad i\in\mathbb{N}_0. \tag{18}$$

Such stochastic processes are the main tool to describe the higher order generalized geometric polynomials considered here. To this end, some properties of these processes will be needed. Let $\tau>0$ be such that

$$\tau<\log(1+1/x). \tag{19}$$

Observe that for any $m\in\mathbb{N}$, $x\geq 0$, and $\tau>0$ satisfying (19), we have

$$\mathbb{E}e^{\tau Z_m(x)}=\sum_{i=0}^{\infty}\binom{-m}{i}\left(-\frac{e^\tau x}{x+1}\right)^i\left(\frac{1}{x+1}\right)^m=\frac{1}{(1-(e^\tau-1)x)^m}<\infty, \tag{20}$$

where $\mathbb{E}$ stands for mathematical expectation. Denote by $\mathcal{E}_\tau$ the set of functions $\phi:\mathbb{N}_0\to\mathbb{R}$ such that

$$|\phi(i)|\leq Ae^{\tau i},\quad i\in\mathbb{N}_0,$$

where $A>0$ and $\tau>0$ satisfies (19). Observe that any polynomial belongs to $\mathcal{E}_\tau$. On the other hand, formula (20) implies that $\mathbb{E}\phi(Z_m(x))$ is finite for any $\phi\in\mathcal{E}_\tau$.

If $\phi \in \mathcal{E}_\tau$, we denote by $\Delta^k \phi$ the usual kth forward difference of ϕ, that is,

$$\Delta^k \phi(l) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} \phi(l+i), \quad k, l \in \mathbb{N}_0.$$

It is well known that if $q_n(x)$ is a polynomial of degree n, then

$$\Delta^k q_n(l) = 0, \quad l \in \mathbb{N}_0, \quad k = n+1, n+2, \ldots \tag{21}$$

Let $m \in \mathbb{N}$, $x \geq 0$, and $\phi \in \mathcal{E}_\tau$. The following crucial formula, shown in [1, Theorem 8.1], computes expectations of functions ϕ acting on the negative binomial process in terms of its forward differences $\Delta^k \phi$ as follows

$$\mathbb{E}\phi(Z_m(x)) = \sum_{i=0}^{\infty} \phi(i) P(Z_m(x) = i) = \sum_{k=0}^{\infty} \binom{m-1+k}{k} \Delta^k \phi(0) x^k. \tag{22}$$

On the other hand, the following auxiliary result will be very useful.

LEMMA 13. *Let $m, \nu \in \mathbb{N}$, $x \geq 0$, and $\phi \in \mathcal{E}_\tau$. Then,*

$$\mathbb{E}\phi(Z_{m+1}(x)) = \frac{1}{m(x+1)} \mathbb{E}\phi(Z_m(x))(Z_m(x) + m) \tag{23}$$

and

$$\mathbb{E}\phi(Z_{m+\nu}(x)) = \sum_{i=0}^{\infty} \mathbb{E}\phi(Z_m(x) + i) \binom{\nu - 1 + i}{i} \left(\frac{x}{x+1}\right)^i \left(\frac{1}{x+1}\right)^\nu. \tag{24}$$

Proof. From (18), we see that

$$P(Z_{m+1}(x) = i) = \frac{i+m}{m(x+1)} P(Z_m(x) = i), \quad i \in \mathbb{N}_0.$$

We therefore have

$$\mathbb{E}\phi(Z_{m+1}(x)) = \sum_{i=0}^{\infty} \phi(i) P(Z_{m+1}(x) = i) = \frac{1}{m(x+1)} \sum_{i=0}^{\infty} \phi(i)(i+m) P(Z_m(x) = i),$$

thus showing (23). As follows from (20),

$$\mathbb{E}e^{\tau Z_{m+\nu}(x)} = \mathbb{E}e^{\tau Z_m(x)} \mathbb{E}e^{\tau Z_\nu(x)}.$$

By the uniqueness theorem for Laplace transforms, this means that the law of $Z_{m+\nu}(x)$ is the same as the law of $Z_m(x) + Z_\nu(x)$, where the random variables $Z_m(x)$ and $Z_\nu(x)$ are supposed to be independent. Hence, we have from (18)

$$\mathbb{E}\phi(Z_{m+\nu}(x)) = \mathbb{E}\phi(Z_m(x) + Z_\nu(x))$$
$$= \sum_{i=0}^{\infty} \mathbb{E}\phi(Z_m(x) + i) \binom{\nu - 1 + i}{i} \left(\frac{x}{x+1}\right)^i \left(\frac{1}{x+1}\right)^\nu.$$

This shows (24) and completes the proof. □

We apply the preceding properties to the problems considered in this paper. We start from the following result which generalizes Nelsen's theorem.

Theorem 14. *Let $m \in \mathbb{N}$ and $x \geq 0$. If $q_n(x)$ is a polynomial of degree n, then*

$$\mathbb{E}q_n(Z_m(x)) = \sum_{i=0}^{\infty} q_n(i)\binom{m-1+i}{i}\left(\frac{x}{x+1}\right)^i\left(\frac{1}{x+1}\right)^m = \sum_{k=0}^{n}\binom{m-1+k}{k}\Delta^k q_n(0)x^k.$$

As a consequence, Nelsen's theorem is true.

Proof. The first statement readily follows from (18), (21), and (22), by choosing $\phi = q_n$. Nelsen's theorem follows by choosing $m = 1$, $x = 1$, and $q_n(x) = (\gamma + x)^n$. The proof is complete. □

As in the previous sections, we assume from now on that $r \in \mathbb{N}_0$, $\alpha, \beta, \gamma, \lambda \in \mathbb{N}$, with $\alpha|\beta$ and $\alpha|\gamma$ and $j = r\beta - r\alpha$. We also assume that $x, t \geq 0$ satisfy

$$|\alpha t| < 1, \qquad (1+\alpha t)^{\beta/\alpha} < 1 + 1/x. \tag{25}$$

The following result is the key tool to give a probabilistic interpretation of higher order generalized geometric polynomials with shifted parameters.

Theorem 15. *With the assumptions in* (25), *we have*

$$\mathbb{E}(1+\alpha t)^{(\beta Z_{\lambda+r}(x)+j+\gamma)/\alpha} = \frac{(1+\alpha t)^{(j+\gamma)/\alpha}}{(1-((1+\alpha t)^{\beta/\alpha}-1)x)^{\lambda+r}} = \sum_{n=0}^{\infty}\mathbb{E}\left(\frac{\beta Z_{\lambda+r}(x)+j+\gamma}{\alpha}\right)_n \alpha^n \frac{t^n}{n!}. \tag{26}$$

As a consequence,

$$w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j) = \mathbb{E}\left(\frac{\beta Z_{\lambda+r}(x)+j+\gamma}{\alpha}\right)_n \alpha^n = \mathbb{E}\left(\beta Z_{\lambda+r}(x)+j+\gamma)/\alpha\right)_n. \tag{27}$$

Proof. The first equality in (26) is shown as in (20). Note that the assumptions in (25) guarantee that all the computations performed make sense. On the other hand, the binomial expansion implies that

$$(1+\alpha t)^{(\beta Z_{\lambda+r}(x)+j+\gamma)/\alpha} = \sum_{n=0}^{\infty}\left(\frac{\beta Z_{\lambda+r}(x)+j+\gamma}{\alpha}\right)_n \alpha^n \frac{t^n}{n!}.$$

Therefore, the second equality in (26) follows by taking expectations in this last expression. Note again that the assumptions in (25) allow us to interchange sum and expectation. Finally, formula (27) is an immediate consequence of definition (5) and identity (26). □

According to this result, higher order generalized geometric polynomials are, up to a constant factor, the expectation of a random descending factorial involving the negative binomial process. In order to obtain an analogue to Theorem 14, we need to compute the *kth* forward difference of the polynomial

$$Q_n(y) = \left(\frac{\beta y + j + \gamma}{\alpha}\right)_n . \tag{28}$$

To this end, recall that

$$(y)_i = \sum_{l=0}^{i} \begin{bmatrix} i \\ l \end{bmatrix} (-1)^{i-l} y^l, \qquad y^i = \sum_{l=0}^{i} \begin{Bmatrix} i \\ l \end{Bmatrix} (y)_l, \quad i \in \mathbb{N}_0. \tag{29}$$

Also, recall that

$$\begin{Bmatrix} i \\ l \end{Bmatrix} = \frac{1}{l!} \Delta^l I_i(0), \quad i \in \mathbb{N}_0, \tag{30}$$

where $I_i(y) = y^i$ is the *ith* monomial function.

Lemma 16. *Let $Q_n(y)$ be as in (28). For any $k = 0, 1, ..., n$, we have*

$$\Delta^k Q_n(0) = k! \sum_{i=k}^{n} \binom{n}{i} \left(\frac{j+\gamma}{\alpha}\right)_{n-i} \sum_{l=k}^{i} (-1)^{i-l} \begin{bmatrix} i \\ l \end{bmatrix} \begin{Bmatrix} l \\ k \end{Bmatrix} \left(\frac{\beta}{\alpha}\right)^l .$$

Proof. Denote by $a = \beta/\alpha$ and by $b = (j + \gamma)/\alpha$. Using the Chu-Vandermonde identity, we get

$$Q_n(y) = n! \binom{ay + b}{n} = n! \sum_{i=0}^{n} \binom{ay}{i} \binom{b}{n-i} = \sum_{i=0}^{n} \binom{n}{i} (ay)_i (b)_{n-i}. \tag{31}$$

For any $i = 0, 1, ..., n$, define the polynomial $p_i(y) = (ay)_i$. From (21), (29), and (30), we have

$$\Delta^k p_i(0) = \sum_{l=k}^{i} (-1)^{i-l} \begin{bmatrix} i \\ l \end{bmatrix} a^l \Delta^k I_l(0) = k! \sum_{l=k}^{i} (-1)^{i-l} \begin{bmatrix} i \\ l \end{bmatrix} \begin{Bmatrix} l \\ k \end{Bmatrix} a^l.$$

This, together with (31), shows the result. □

Observe that if $\alpha = \beta$, Lemma 16 takes on the simple form

$$\Delta^k Q_n(0) = (n)_k \left(\frac{j+\gamma}{\alpha}\right)_{n-k}, \quad k = 0, 1, ..., n.$$

We are in a position to compute $w_n^{(\lambda+r)}(x; \alpha, \beta, \gamma + j)$ in terms of a finite sum.

Theorem 17. *With the preceding notations, we have*

$$\begin{aligned} w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j) &= \\ &\alpha^n \sum_{i=0}^{\infty} \left(\frac{\beta i+j+\gamma}{\alpha}\right)_n \binom{\lambda+r-1+i}{i} \left(\frac{x}{x+1}\right)^i \left(\frac{1}{x+1}\right)^{\lambda+r} \\ &= \alpha^n \sum_{k=0}^{n} \binom{m-1+k}{k} \Delta^k Q_n(0) x^k, \end{aligned}$$

where $\Delta^k Q_n(0)$ is computed in Lemma 16.

Proof. It suffices to apply Theorem 14 with $q_n(y) = Q_n(y)$, as defined in (28), as well as formula (27). □

The properties given in Lemma 13 are used to establish the following two results.

Theorem 18. *The following relation holds true:*

$$\begin{aligned} &\beta(x+1)(\lambda+r)w_n^{(\lambda+r+1)}(x;\alpha,\beta,\gamma+j) = \\ &\qquad w_{n+1}^{(\lambda+r)}(x;\alpha,\beta,\gamma+j+\alpha) + (\beta\lambda+\alpha(r-1)-\gamma)w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j). \end{aligned}$$

Proof. Denote by $a = \beta/\alpha$, $b = (j+\gamma)/\alpha$, and $m = \lambda + r$. Choosing $\phi(y) = (ay+b)_n$ in (23), we get

$$\begin{aligned} ma(x+1)\mathbb{E}(aZ_{m+1}(x)+b)_n &= \mathbb{E}(aZ_m(x)+b)_n(aZ_m(x)+ma) \\ &= \mathbb{E}(aZ_m(x)+b)_n(aZ_m(x)+b+1+ma-b-1) \\ &= \mathbb{E}(aZ_m(x)+b+1)_{n+1} + (ma-b-1)\mathbb{E}(aZ_m(x)+b)_n. \end{aligned} \tag{32}$$

Multiplying both sides of (32) by $(m-1)!\alpha^{n+1}$, the result follows from the first equality in (27). □

Theorem 19. *For any $\nu \in \mathbb{N}$, we have*

$$\begin{aligned} w_n^{(\lambda+r+\nu)}(x;\alpha,\beta,\gamma+j) = (\lambda+r+\nu-1)_\nu &\sum_{i=0}^{\infty} w_n^{(\lambda+r)}(x;\alpha,\beta,\gamma+j+\beta i) \\ &\times \binom{\nu-1+i}{i} \left(\frac{x}{x+1}\right)^i \left(\frac{1}{x+1}\right)^{\nu}. \end{aligned}$$

Proof. We use the same notations as in the proof of Theorem 18. Choosing $\phi(y) = (ay+b)_n$ in (24), we obtain

$$\mathbb{E}(aZ_{m+\nu}(x)+b)_n = \sum_{i=0}^{\infty} \mathbb{E}(aZ_m(x)+aj+b)_n \binom{\nu-1+i}{i} \left(\frac{x}{x+1}\right)^i \left(\frac{1}{x+1}\right)^{\nu}.$$

Multiplying both sides of this equation by $(\lambda+r+\nu-1)_\nu \alpha^n$ and recalling the first equality in (27), we get the result. □

Acknowledgements. The authors would like to thank the referees for their careful reading of the manuscript, and for their comments and suggestions, which greatly improved the final outcome. The first author was supported by research grant PGC2018-097621-B-100.

References

1. José A. Adell, Differential calculus for linear operators represented by finite signed measures and applications, *Acta Mathematica Hungarica* **138**(1) (2013), 44–82.
2. Connor Ahlbach, Jeremy Usatine, and Nicholas Pippenger, Barred Preferential Arrangements, *The Electronic Journal of Combinatorics* **20**(2) (2013), P55.
3. Hacène Belbachir and Yahia Djemmada, Congruence for geometric polynomials, *Les Annales Rectis* **6** (2019), 37–44.
4. Beáta Bényi and Toshiki Matsusaka, On the Combinatorics of Symmetrized Poly-Bernoulli Numbers, *The Electronic Journal of Combinatorics,* **28**(1) (2021), P1.47.
5. Khristo N. Boyadzhiev, A series transformation formula and related polynomials, *International Journal of Mathematics and Mathematical Sciences,* **2005**(23) (2005), 3849–3866.
6. Khristo N. Boyadzhiev and Ayhan Dil, Geometric polynomials: properties and applications to series with zeta values, *Analysis Mathematica,* **42**(3) (2016), 203–224.
7. Andrei Z. Broder, The r-Stirling numbers, *Discrete Mathematics* **49**(3) (1984), 241–259.
8. Cristina B. Corcino, Roberto B. Corcino, Bayram Çekim, Levent Kargin, and Sithembele Nkonkobe, A second type of higher order generalized geometric polynomials and higher order generalized Euler polynomials, *Quaestiones Mathematicae,* (2020), https://doi.org/10.2989/16073606.2020.1848937
9. Roberto B. Corcino, Some theorems on generalized Stirling numbers, *Ars Combinatoria* **60** (2001), 273–286.
10. Roberto B. Corcino and Cristina B Corcino, On generalized Bell polynomials, *Discrete Dynamics in Nature and Society* **3** (2011), 623456.
11. Roberto B. Corcino, Leetsch C. Hsu, and Evelyn L. Tan, Combinatorial and statistical applications of generalized Stirling numbers, *Journal of Mathematical Research and Exposition – Chinese Edition* **21**(3) (2001), 337–343.
12. Martha E. Dasef and Steven M. Kautz, Some sums of some importance, *The College Mathematics Journal* **28** (1997), 52–55.
13. Toka Diagana and Hamadoun Maïga, Some new identities and congruences for Fubini numbers, *Journal of Number Theory* **173** (2017), 547–569.
14. Ayhan Dil and Veli Kurt, Investigating geometric and exponential polynomials with Euler-Seidel matrices, *Journal of Integer Sequences* **14**(4) (2011), A 11.4.6.

15. Leonhard Euler, *Institutiones calculi differentialis cum ejus usu in analysi finitorum ac doctrina serierum,* Academiae Imperialis Scientiarum, St. Petersburg, 1755.
16. Philippe Flajolet and Robert Sedgewick, *Analytic combinatorics,* Cambridge University Press, New York, 2009.
17. Oliver A. Gross, Preferential arrangements, *The American Mathematical Monthly* **69**(1) (1962), 4–8.
18. Meigu Guan, Weixuan Li, Edward T.H. Wang, Ira Gessel, T. Ferguson, C. Melolidakis, Roger B. Nelsen, and Ion Cucurezeanu, Elementary Problems: E3059-E3063, *The American Mathematical Monthly* **91**(9) (1984), 580–581.
19. Leetsch C. Hsu and Peter J. Shiue, A unified approach to generalized Stirling numbers, *Advances in Applied Mathematics* **20**(3) (1998), 366–384.
20. Levent Kargin, Some formulae for products of geometric polynomials with applications, *Journal of Integer Sequences* **20** (2017), A 17.4.4.
21. Levent Kargin and Bayram Cekim, Higher order generalized geometric polynomials, *Turkish Journal of Mathematics* **42**(3) (2018), 887–903.
22. Levent Kargin and Mehmet Cenkci, Recurrences and congruences for higher order geometric polynomials and related numbers 2019, arXiv preprint arXiv:1910.13248.
23. Levent Kargin and Roberto B. Corcino, Generalization of Mellin derivative and its applications *Integral Transforms and Special Functions* **27**(8) (2016), 620–631.
24. Elliott Mendelson, Races with ties, *Mathematics Magazine* **55**(3) (1982), 170–175.
25. Miloud Mihoubi and Said Taharbouchet, Identities and congruences involving the geometric polynomials, *Miskolc Mathematical Notes* **20**(1) (2019), 395–408.
26. Roger B. Nelsen, Donald E. Knuth, J.C. Binz, and Gordon Williams, E3062, *The American Mathematical Monthly* **94**(4) (1987), 376–377.
27. Sithembele Nkonkobe, *A combinatorial analysis of barred preferential arrangements,* Ph.D. Thesis, Rhodes University, South Africa, 2016.
28. Sithembele Nkonkobe, Beáta Bényi, Roberto B. Corcino, and Cristina B. Corcino, A combinatorial analysis of higher order generalised geometric polynomials: A generalisation of barred preferential arrangements, *Discrete Mathematics* **343**(3) (2020), 111729.
29. Sithembele Nkonkobe and Venkat Murali, A study of a family of generating functions of Nelsen-Schmidt type and some identities on restricted barred preferential arrangements, *Discrete Mathematics* **340**(5) (2017), 1122–1128.
30. Sithembele Nkonkobe, Venkat Murali, and Beáta Bényi, Generalised Barred Preferential Arrangements, 2019, arXiv preprint arXiv:1907.08944.
31. Nicholas Pippenger, The hypercube of resistors, asymptotic expansions, and preferential arrangements, *Mathematics Magazine* **83**(5) (2010), 331–346.
32. Neil J.A. Sloane, The On-line Encyclopedia of Integer Sequences, https://oeis.org.
33. Stephen M. Tanny, On some numbers related to the Bell numbers, *Canadian Mathematical Bulletin* **17**(5) (1975), 733–738.

Received 22 February, 2021 and in revised form 19 December, 2021.