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Combinatorial approach of unified Apostol-type polynomials using α-distanced words

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ABSTRACT

We introduce and enumerate weighted α-distanced words. We show their connection to the unified Apostol-type polynomials and provide combinatorial proofs of certain identities.

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1. Introduction

A word over an alphabet $\mathcal{A}$ is a finite sequence of elements from $\mathcal{A}$ (see Lothaire [25]). Combinatorics on words is a comparatively new area of discrete mathematics. Berstel and Perrin [4] presented an overview of the historical roots of this area. Words are appropriate tools for decoding several objects and its theory has found applications in algebra, geometry, graph theory, and automata theory. Enumerations of different classes of words lead to relations of certain polynomials. One beautiful example arises in the works of Kucukoglu in that the relationship between k-ary Lyndon words and the Apostol-type numbers and polynomials are revealed. See [22,23] and the references there. In this paper we present another connection of the Apostol-type polynomials with other types of words, the weighted α-distanced words that we introduce. Actually, our goal is to show one possible combinatorial interpretation of the unified Apostol-type polynomials.

One of the most famous number sequences are the Bernoulli numbers. Euler and Genocchi numbers are closely related sequences. All of these numbers and polynomials play role in several branches of mathematics such as combinatorics, umbral calculus, analytic number theory and other related fields. Bernoulli ($B_n(x)$), Euler ($E_n(x)$) and Genocchi ($G_n(x)$) polynomials are defined as

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, \quad \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1}e^{xt}, \quad \text{and} \quad \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt}.$$

The literature on these polynomials and their generalizations is enormous, reaching in many different directions. Luo [26] called the following polynomials *Apostol–Bernoulli polynomials* because Apostol [1] defined them first in order to evaluate the Lerch transcendent (also called Lerch zeta function), $\Phi(\lambda, s, a) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+a)^s}$, for negative integer values s:

$$\sum_{n=0}^{\infty} B_n(x;\lambda)\frac{t^n}{n!} = \frac{t}{\lambda e^t-1}e^{xt}, \quad (|t+\ln\lambda| < 2\pi, \lambda \in \mathbb{C}).$$

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The Apostol-type generalization of the other related polynomials (Euler, Genocchi, etc.) were similarly introduced and then studied. The common properties of these polynomials are revealed by the so called unified polynomials. Ozden [33] introduced a polynomial class that includes Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials and Özarslan [32] extended it for the higher order polynomials as

$$\sum_{n=0}^{\infty} P_{n,\beta}^{(\alpha)}(x;k,a,b)\frac{t^n}{n!} = \left(\frac{2^{1-k}t^k}{\beta^b e^t - a^b}\right)^{\alpha} e^{xt},$$

where $|t + b\ln\left(\frac{\beta}{a}\right)| < 2\pi$, $k \in \mathbb{N}$, $a, b \in \mathbb{R}^+$, and $\alpha, \beta \in \mathbb{C}$.

A similar unification including also the Fubini polynomials is given by Kurt [24] as follows:

$$\sum_{n=0}^{\infty} F_n^{\alpha}(x,y;a,b,\beta,k)\frac{t^n}{n!} = \left(\frac{t^k}{1 - y(\beta^b e^t - a^b)}\right)^{\alpha} e^{xt}, \tag{1}$$

where $\left|t + \ln\frac{y\beta^b}{1+ya^b}\right| < 2\pi$, $a, b \in \mathbb{R}^+$, $x, y \in \mathbb{R}$, $k, \alpha \in \mathbb{N}_0$.

Note that setting $k = \alpha = 1$, $y = -2$, $a = b = 1$ and $\beta = \frac{\lambda}{2}$ we obtain the Apostol–Bernoulli polynomials, $k = 0$, $\alpha = 1$, $y = -\frac{1}{2}$, $a = b = 1$ and $\beta = \lambda$, the Apostol–Euler-polynomials, $k = \alpha = 1$, $y = -\frac{1}{2}$, $a = b = 1$ and $\beta = \lambda$ the Apostol–Genocchi polynomials.

The study of the degenerate version of these polynomials has been started by Carlitz [8,9] by the investigation of the *degenerate Bernoulli polynomials*, defined as

$$\sum_{n=0}^{\infty} B_n(x|\lambda)\frac{t^n}{n!} = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}}.$$

Note that by taking the limit of the degenerate polynomials, $B_n(x|\lambda)$, we obtain the classical Bernoulli polynomials:

$$\lim_{\lambda\to 0}\left(\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}(1+\lambda t)^{\frac{x}{\lambda}}\right) = \frac{t}{e^t - 1}e^{xt}.$$

Kurt [24] defined and studied also the degenerate version of his unified Apostol-type polynomials. We consider in this paper the degenerate case because in some sense this is a more general model, at least from the combinatorial point of view. The arguments that we formulate in the proofs involving the degenerate polynomials can be easily modified for the non-degenerate case.

Precisely, for a, j, b, λ real numbers and d, α non-negative integers, let $\mathcal{F}_{n,x}^{\lambda}(\alpha,1,\gamma,j,a,b,d)$ denote the *degenerate unified Apostol-type Bernoulli, Euler, Genocchi, and Fubini-polynomials of order* α that is defined by the following generating function:

$$\sum_{n=0}^{\infty} \mathcal{F}_{n,x}^{\lambda}(\alpha,1,\gamma,j,a,b,d)\frac{t^n}{n!} = \left(\frac{t^d}{1 - x(b^j(1+\alpha t)^{\frac{1}{\alpha}}) - a^j}\right)^{\lambda}(1+\alpha t)^{\frac{\gamma}{\alpha}}, \tag{2}$$

where x and γ denote indeterminates.

Note that we used by recalling the definition a different notation as in the original paper which may be first confusing. However, for our purposes in this paper we needed this slight modification.

One of the advantages of the study of unified polynomials is that the results include many special cases. The special polynomials play important role in different branches of mathematics and it is fascinating that each line of research reveals their properties from another point of view. Such a special point of view comes from combinatorics. In this paper we show a possible combinatorial approach of the unified polynomials introduced by Kurt.

For this reason, we will study the following generalization of the degenerate unified Apostol-type polynomials (2) introduced by Kurt [24]. Let a, b, β, γ, λ, α non-negative integers such that both β and γ are multiplicatives of α. Then let $\mathcal{F}_{n,x}^{\lambda}(\alpha,\beta,\gamma,a,b)$ denote the polynomials defined by the following generating function:

$$\sum_{n=0}^{\infty} \mathcal{F}_{n,x}^{\lambda}(\alpha,\beta,\gamma,a,b)\frac{t^n}{n!} = \frac{(1+\alpha t)^{\frac{\gamma}{\alpha}}}{[1 - x[b(1+\alpha t)^{\frac{\beta}{\alpha}} - a]]^{\lambda}}. \tag{3}$$

Note that the polynomials obtained by setting $\beta = 1$ in (3) and the polynomials obtained by setting $d = 0$ and $j = 1$ in (2) are identical.

Setting the parameters as $(\alpha,\beta,\gamma,\lambda,a,b,x) = (0,1,0,\lambda,1,1,x)$ we obtain Fubini polynomials [5,6]. Tanny [35] defines the *Fubini polynomials* $F_n(x)$ by the explicit formula

$$F_n(x) = \sum_{k=0}^{n} k!\left\{{n \atop k}\right\}x^k,$$

with the exponential generating function

$$\sum_{n=0}^{\infty} F_n(x)\frac{t^n}{n!} = \frac{1}{1-x(e^t-1)}.$$

Fubini polynomials are also called *geometric polynomials* (and in this line of research are denoted by $w_n(x)$) because of their relation to geometric series [5]

$$(xD)^m\left\{\frac{1}{1-x}\right\} = \sum_{k=0}^{\infty} k^m x^k = \frac{1}{1-x} w_n\left(\frac{x}{1-x}\right), \quad |x|<1.$$

Finally, we list some more special cases that we obtain when we set special values of the parameters in $\mathcal{F}_{n,x}^{\lambda}(\alpha,\beta,\gamma,a,b)$.

- The higher order generalized geometric polynomials studied in [16,30] are the special case $(\alpha,\beta,\gamma,\lambda,a,b,x) = (\alpha,\beta,\gamma,\lambda,1,1,x)$.
- The Fubini type polynomials studied in [29] are $(\alpha,\beta,\gamma,\lambda,a,b,x) = (0,1,0,\lambda,\frac{1-\sqrt{2}}{\sqrt{2}},\frac{1}{\sqrt{2}},x)$.
- The higher order generalized Euler polynomials discussed in [13] are the special case $(\alpha,\beta,\gamma,\lambda,a,b,x) = (\alpha,\beta,\gamma,\lambda,1,1,-\frac{1}{2})$.
- The two variable higher order Fubini polynomials discussed in [20] are

  $$(\alpha,\beta,\gamma,\lambda,a,b,x) = (0,1,\gamma,\lambda,1,1,x).$$

- The generalized ordered Bell polynomials studied in [14] are

  $$(\alpha,\beta,\gamma,\lambda,a,b,x) = (0,\beta,\gamma,\frac{\beta'+\gamma'}{\beta'},\frac{\beta' q}{\beta},\frac{\beta' q}{\beta},1).$$

- The degenerate ordered Bell polynomials discussed in [18] are

  $$(\alpha,\beta,\gamma,\lambda,a,b,x) = (\alpha,1,\gamma,\lambda,1,1,1).$$

We consider first the notion of weighted α-distanced words which are at the heart of our combinatorial model. In the next section we investigate an appropriate generalization of (3) and present elementary proofs of some identities. Finally, we collect some remarks and applications of our results.

Throughout this paper, we always make use of the following notation, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_0$ denotes the set of non-negative integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. As usual, $(x|\alpha)_n$ denotes the generalized factorial with increment α (see [9]):

$$(x|\alpha)_n = x(x-\alpha)\cdots(x-(n-1)\alpha).$$

2. α-distanced words

An ordered partition of a set $[n]=\{1,2,\ldots,n\}$ is a linear arrangement of subsets (called blocks) of the elements $[n]$. The number of ordered partitions are given by the Fubini numbers, F_n, (also called ordered Bell numbers, preferential arrangement numbers, geometric numbers) (see [10,11,31]). The generating function of F_n is given as

$$\sum_{n=0}^{\infty} F_n\frac{t^n}{n!} = \frac{1}{2-e^t}.$$

In a natural way ordered partition of $[n]=\{1,2,\ldots,n\}$ into k blocks can be encoded by a word of length n with entries of $[k]$ such that each of the entries arises at least once (see [21]). For instance, the ordered partition $\{6\}\{1,4,5\}\{3,7\}\{2,8\}$ is encoded by the word 24322134. The set of all such words are the surjections $[n]\to[k]$. Clearly, the number of these surjections is $k!\left\{{n \atop k}\right\}$, where $\left\{{n \atop k}\right\}$ denotes the Stirling number of the second kind that counts partitions of $[n]$ into k non-empty blocks. Hence, Fubini numbers, $\sum k!\left\{{n \atop k}\right\}$, enumerate words over the alphabet $\{1,2,\ldots,k\}$ (for a $k\geq 1$) of length n such that all elements of the alphabet arise.

We now consider words over a specific alphabet with some restrictions for the common occurrences of letters in a word. Given k and α positive integers and β, γ multiplicatives of α, let $\mathcal{A}(k,\beta,\gamma)$ denote an alphabet containing $k\beta+\gamma$ letters. We assume that in the alphabet there are α similar variants of a symbol, as it is the case with the variants of letters having different accents in many languages (this property explains the condition on the integers β and γ to be divisible by α).

Further, we assume that the symbols in $[\beta]$ are available in k different sizes whereas the symbols in $[\gamma]$ have all a standard size. We call β letters having the standard size as the first β-group, the second β letters having another size as the second β-group etc. We call the α similar variants of a symbol as α-*block*

Example 1. Let $\mathcal{A}$ be an alphabet containing letters and numbers, where both can have an accent and the letters are available in three different sizes. Then, $\alpha = 2$, the two types of each symbol: without and with accent and $k = 3$. Choosing $\beta = 6$ and $\gamma = 8$, the following list is an example for an alphabet with the above properties.

$\mathcal{A} = \{$ 1, 1́, 2, 2́, 3, 3́, 4, 4́, a, á, e, é, o, ó, a, á, e, é, o, ó, a, á, e, é, o, ó, $\}$.

The alphabet is built up with 2-blocks, as for instance a, á.

As next, we introduce the notion of α-distanced words. We emphasized that in the alphabet there are α similar variants of each symbol. During a coding and decoding process such words could play an important role in that only one variant of each symbol can occur. We consider the class of such words.

More precisely, we say that a word u is *α-distanced* if from each α-block only one letter occurs in u. A 2-distanced word over the alphabet of our example is a word that contains a symbol either with an accent or without an accent, but not both.

Note that the above condition implies the requirement that each letter is distinct in u.

Let $\mathcal{W}_n(\mathcal{A};\alpha)$ denote the set of α-distanced words of length n over the alphabet $\mathcal{A}(k,\beta,\gamma)$. (We omit the parameters k, β, γ in order to keep the notation simple.)

Consider now a word $u \in \mathcal{W}_n(\mathcal{A};\alpha)$ and let i be the number of the k available distinct β-groups from that a letter occurs in u ($i \le k$). We assign the weight, $w(u)$, to the word u as $w(u) = b^i(b-a)^{k-i}$, where a and b are indeterminates.

Example 2. The word $z =$ a1oeá is not a 2-distanced word over the alphabet $\mathcal{A}$ (given in Example 1) because a and á both occur. The word $u =$ 1oáae is a 2-distanced word with weight $b^3(b-a)^0$, since all of the 3 available different sizes of the symbols are represented in the word u.

We consider the set of weighted α-distanced words and show some properties. Given k, α, β and γ positive integers with $\alpha|\beta$ and $\alpha|\gamma$, let $G_n^k(\alpha,\beta,\gamma,a,b)$ denote the total number of weighted α-distanced words of length n over the alphabet $\mathcal{A}(k,\beta,\gamma)$:

$$\sum_{z\in\mathcal{W}_n(\mathcal{A};\alpha)} w(z) = G_n^k(\alpha,\beta,\gamma,a,b).$$

Theorem 3. *For n, k, α, β, γ positive integers with $k \le n$, $\alpha|\beta$ and $\alpha|\gamma$, a, b indeterminates, we have*

$$G_n^k(\alpha,\beta,\gamma,a,b) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}b^i a^{k-i}(\beta i+\gamma|\alpha)_n. \tag{4}$$

Proof. We start with an example. Consider the words that contain letters only from $[\gamma]$ (where symbols from $[\gamma]$ have standard size). The number of such words is $(\gamma|\alpha)_n$ and since each such word has weight $(b-a)^k$ the sum of weights of all these words is

$$(\gamma|\alpha)_n(b-a)^k = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}b^i a^{k-i}(\gamma|\alpha)_n$$

by the binomial theorem.

Note that $(\beta+\gamma|\alpha)_n$ counts words in that *at most* one given β-group occurs. In general, the number of words in that *at most* i given β-groups occur is $(\beta i+\gamma|\alpha)$.

In order to handle the general case we introduce some notation. Let $z_{|\beta|=j}$ denote a word that contains exactly j of the β-groups and let $|z_{|\beta|=j}|$ denote the number of these words. Further, let $z_{|\beta|\le j}$ denote a word that contains at most j of the β-groups and $|z_{|\beta|\le j}|$ the number of such words.

Therefore,

$$|z_{|\beta|\le j}| = (\beta j+\gamma|\alpha)_n = \sum_{\ell=0}^{j}\binom{j}{\ell}|z_{|\beta|=\ell}|.$$

By the binomial theorem we have,

$$\begin{aligned}\sum_{z\in\mathcal{W}_n(\mathcal{A};\alpha)} w(z) &= \sum_{i=0}^{k}\binom{k}{i}b^i(b-a)^{k-i}|z_{|\beta|=i}|\\ &= \sum_{i=0}^{k}\sum_{j=i}^{k}(-1)^{k-j}\binom{k}{j}\binom{j}{i}b^j a^{k-j}|z_{|\beta|=i}|\end{aligned}$$

$$= \sum_{j=0}^{k} (-1)^{k-j} b^j a^{k-j} \left(\sum_{i=0}^{j} \binom{j}{i} z_{|\beta|=i|} \right)$$
$$= \sum_{j=0}^{k} (-1)^{k-j} b^j a^{k-j} |z_{|\beta|\le j}|$$
$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} b^j a^{k-j} (j\beta + \gamma|\alpha)_n. \quad \square$$

A special case of (4) is Theorem 1 of [12].

We now prove two recurrences that $G_n^k(\alpha, \beta, \gamma, a, b)$ fulfills.

Theorem 4. *For n, k, α, β, γ positive integers with $k \le n$, $\alpha|\beta$ and $\alpha|\gamma$, a, b indeterminates, it holds*

$$G_{n+1}^k(\alpha, \beta, \gamma, a, b) = \gamma G_n^k(\alpha, \beta, \gamma - \alpha, a, b) + kb\beta G_n^{k-1}(\alpha, \beta, \gamma + \beta - \alpha, a, b).$$

Proof. Let $u \in \mathcal{W}_{n+1}(\mathcal{A}; \alpha)$. Consider the first entry u_1. It is a letter from $[\gamma]$ or a letter from one of the β-groups. In the first case for the remaining letters $u_2 \cdots u_{n+1}$ there are $k\beta + \gamma - \alpha$ letters available and the weight of the entire word is independent of u_1. Hence, the sum of weights of all such words is $\gamma G_n^k(\alpha, \beta, \gamma - \alpha, a, b)$. In the second case there are again $k\beta + \gamma - \alpha$ letters available for the remaining word, however, the weight will depend on the choice of u_1. We count first the weight of $u_2 \cdots u_{n+1}$ such that the β-group of u_1 is ignored, i.e., as a word of the alphabet $\mathcal{A}(k-1, \beta, \gamma + \beta - \alpha)$. Then, this weight has to be multiplied by k for the choice of the particular β-group of u_1, by β for the particular value of u_1, and by b for the weight that we ignored previously. $\square$

Theorem 5. *For n, k, α, β, γ positive integers with $k \le n$, $\alpha|\beta$ and $\alpha|\gamma$, a, b indeterminates the recurrence relation*

$$G_{n+1}^k(\alpha, \beta, \gamma, a, b) = (\gamma + k\beta - n\alpha) G_n^k(\alpha, \beta, \gamma, a, b) + ak\beta G_n^{k-1}(\alpha, \beta, \gamma, a, b)$$

holds.

Proof. Take a word $u \in \mathcal{W}_n(\mathcal{A}; \alpha)$ and add an $(n+1)$th entry, u_{n+1}, at the end to obtain u'. This entry can be chosen in $\gamma + k\beta - n\alpha$ ways since there are $k\beta + \gamma$ letters in the alphabet, but u_{n+1} cannot be chosen from any of the α-blocks of the previously occurred n letters. The weight of some words will be changed by adding this new entry. Namely, if u_{n+1} is chosen from a β-group that has not occurred in u. Hence, in the sum $(\gamma + k\beta - n\alpha) G_n^k(\alpha, \beta, \gamma, a, b)$ the weight of a word in that i β-groups occur and u_{n+1} is unique in its β-group is (incorrectly) $b^{i-1}(b-a)^{k-i+1}$ instead of the correct weight $b^i(b-a)^{k-i}$. The difference is $b^{i-1}(b-a)^{k-i+1} - b^i(b-a)^{k-i} = -ab^{i-1}(b-a)^{k-i}$.

We take account of this difference as follows. We take words of length n with $k-1$ different available β-groups. The sum of the weights of all such words is $G_n^{k-1}(\alpha, \beta, \gamma, a, b)$. Note that in this case each word where $i-1$ of the β-groups occur has the weight $b^{i-1}(b-a)^{k-i}$. We multiply the weight by a in order to get the necessary difference, and by $k\beta$, the number of different possible values of u_{n+1}. $\square$

3. Further results

In this section we present a possible combinatorial model for the numbers $\mathcal{F}_{n,x}^{\lambda}(\alpha, \beta, \gamma, a, b)$. We have to augment the model of $G_n^k(\alpha, \beta, \gamma, a, b)$ by appropriate interpretations of x and λ. First, let x be an indeterminate and assign to a word u over the alphabet $\mathcal{A}(k, \beta, \gamma)$ the slight modified weight $w'(u) = b^i(b-a)^{k-i}x^k$. Hence, the exponent of x denotes the number of distinct β-groups in the alphabet. (With the terminology of our running example, the possible number of sizes that could arise in a word.) The role of the parameter λ seems to be independent of the word itself. Consider k dots and $\lambda - 1$ bars ordered linearly. Such an arrangement can be decoded by listing the relative positions of the bars. For instance, $(\| \cdot \cdot \cdot | \cdot \| \cdot \cdot)$ is decoded by the sequence $(0, 0, 3, 4, 4)$, where 0 is the number of dots to the left of the first and second bar, 3 is the number of dots to the left of the third bar, and 4 is the number of the dots to the left of the fourth and fifth bar. A sequence that decodes k dots and $\lambda - 1$ bars this way is of length $\lambda - 1$ and have entries from the set $\{0, 1, 2, \ldots, k\}$. Note that the sequence is non-decreasing. We are now ready to present the interpretation of $\mathcal{F}_{n,x}^{\lambda}(\alpha, \beta, \gamma, a, b)$.

Let $\mathcal{US}(n, k, \lambda, \alpha, \beta, \gamma)$ denote the set of all pairs (u, s) where u is an α-distanced word of length n over $\mathcal{A}(k, \beta, \gamma)$ and s is a non-decreasing sequence with values from $[k+1]$ of length $\lambda - 1$. We associate the weight to a pair (u, s) where i $(i \le k)$ is the number of distinct β-groups from that a letter occurs in the word u as

$$w(u, s) = b^i(b-a)^{k-i}x^k.$$

Example 6. For instance, $(u, s) = (1oáae, \{1, 1, 2, 4, 4\})$ is a valid pair where $\lambda = 6$ and the underlying alphabet of u is the one given in Example 1.

From (3) we have by the binomial theorem

$$\frac{(1+\alpha t)^{\frac{\gamma}{\alpha}}}{[1-x[b(1+\alpha t)^{\frac{\beta}{\alpha}}-a]]^{\lambda}}=(1+\alpha t)^{\frac{\gamma}{\alpha}}\sum_{k=0}^{\infty}\binom{k+\lambda-1}{k}x^k[b(1+\alpha t)^{\frac{\beta}{\alpha}}-a]^k.$$

Then it immediately follows that

$$\begin{aligned}\mathcal{F}^{\lambda}_{n,x}(\alpha,\beta,\gamma,a,b)&=\sum_{k=0}^{n}\binom{k+\lambda-1}{k}x^k\sum_{i=0}^{k}\binom{k}{i}b^i a^{k-i}(-1)^{k-i}(\beta i+\gamma|\alpha)_n\\&=\sum_{k=0}^{n}\binom{k+\lambda-1}{k}x^k G^k_n(\alpha,\beta,\gamma,a,b).\end{aligned}\tag{5}$$

Theorem 7. *The weighted sum of all pairs in the set $\mathcal{US}(n,k,\lambda,\alpha,\beta,\gamma)$ is given by $\mathcal{F}^{\lambda}_{n,x}(\alpha,\beta,\gamma,a,b)$, i.e.,*

$$\sum_{k=0}^{n}\left[\sum_{(u,s)\in\mathcal{US}(n,k,\lambda,\alpha,\beta,\gamma)}w(u,s)\right]=\mathcal{F}^{\lambda}_{n,x}(\alpha,\beta,\gamma,a,b).$$

Proof. For n, k, α, β, γ, λ positive integers with $k\le n$, $\alpha|\beta$, $\alpha|\gamma$, and a, b, x indeterminates, (5) can be interpreted as giving number of weighted sums of α-distanced words, i.e., it gives the sum of all weighted pairs from the set $\mathcal{US}(n,k,\lambda,\alpha,\beta,\gamma)$. The weight of a pair (u,s) depends only on the word u, hence, for a fixed k it is $x^kG^k_n(\alpha,\beta,\gamma,a,b)$. With a fixed $u\in\mathcal{W}_n(\mathcal{A};\alpha)$ there are $\binom{k+\lambda-1}{k}$ possible sequences s that lead to a valid pair (u,s) from the set $\mathcal{US}(n,k,\lambda,\alpha,\beta,\gamma)$. Thus, summing over k we have $\mathcal{F}^{\lambda}_{n,x}(\alpha,\beta,\gamma,a,b)=\sum_{k=0}^{n}\left[\sum_{(u,s)\in\mathcal{US}(n,k,\lambda,\alpha,\beta,\gamma)}w(u,s)\right]$. □

Next, we use this combinatorial interpretation to prove some identities.

Theorem 8. *For n, k, α, β, γ, λ positive integers with $k\le n$, $\alpha|\beta$, $\alpha|\gamma$, and a, b, x indeterminates, we have*

$$\begin{aligned}\mathcal{F}^{\lambda-1}_{n,x}(\alpha,\beta,\gamma,a,b)=&(1+xa)\mathcal{F}^{\lambda}_{n,x}(\alpha,\beta,\gamma,a,b)\\&-xb\sum_{i=0}^{n}\binom{n}{i}\mathcal{F}^{\lambda}_{i,x}(\alpha,\beta,\gamma,a,b)(\beta|\alpha)_{n-i}.\end{aligned}$$

Proof. In this proof we make use of the "bar-dot" representation of the sequence s in a pair (u,s). Let us denote this representation by $\bar{s}$. We rewrite the above identity in the following form. We write $\mathcal{F}^{\lambda}_{n,x}$ short for $\mathcal{F}^{\lambda}_{n,x}(\alpha,\beta,\gamma,a,b)$.

$$\mathcal{F}^{\lambda}_{n,x}-\mathcal{F}^{\lambda-1}_{n,x}=x\left[b\sum_{i=0}^{n-1}\binom{n}{i}\mathcal{F}^{\lambda}_{i,x}(\beta|\alpha)_{n-i}+(b-a)\mathcal{F}^{\lambda}_{n,x}\right].$$

The left hand side counts those (u,s) pairs where in $\bar{s}$ the first and second bars do not follow directly each other, i.e., the first and second entries in s are not the same. This is clear, since from the number of all pairs we substitute the number of words, with one less bar. Those with one less bar are the same as those with $\lambda-1$ bars but the first two bars glued together. We count now these words in another way, proving with that the identity.

First, choose i ($i>0$) from the n positions and construct words of length i, denoted by $u|_i$, with sets s. Note that $\mathcal{F}^{\lambda}_{i,x}$ counts words with $\lambda-1$ bars and certain number of β-groups and γ group. We take such a pair $(u|_i,s)$. Now, we insert into the underlying alphabet a new β-group and add to the remaining positions letters from this β-group. Because of this new group, the weight of the so obtained (u,s) pair is the same as the weight of $(u|_i,s)$ multiplied by xb, i.e., $w(u,s)=xbw(u|_i,s)$. Further, the $\bar{s}$ has to contain one more dot, which we place directly next to the first bar. The term $xb\sum_{i=0}^{n-1}\binom{n}{i}\mathcal{F}^{\lambda}_{i,x}(\beta|\alpha)_{n-i}$ counts the weight of such words.

However, if we augment the underlying alphabet with a new β-group even if none of the entry of the word u is from this group, it influences the weight of the pair (u,s). Namely, $w(u,s)=x(b-a)w(u,s)$. Similarly as before, $\bar{s}$ has to contain one more dot, which we place directly next to the first bar. The term $x(b-a)\mathcal{F}^{\lambda}_{n,x}$ counts the weight of those words where no elements from the new β-group occur. □

We define now a product $\star$ on (u,s) pairs. Let u be a word over the alphabet $\mathcal{A}_1=\mathcal{A}(k_1,\beta,\gamma_1)$ and v over the alphabet $\mathcal{A}_2=\mathcal{A}(k_2,\beta,\gamma_2)$. Let s be a no-decreasing sequence of size λ_1-1 from the values $[k_1+1]$ and t be a sequence of size λ_2-1 from the values k_2+1. We define the *star-product* $(u,s)\star(v,t)=(z,r)$, where

- z is the word that we obtain by writing the word u and v after each other and
- r is the sequence $s \cup \{k_1+1\} \cup t^{\rightarrow}$, where $t^{\rightarrow}$ is the sequence of values in t shifted by k_1+1.

Example 9. Let $k_1 = 3$, $\lambda_1 = 6$, $k_2 = 5$ and $\lambda_2 = 7$ and $s = (1,1,2,4,4)$ and $t = (1,4,4,4,5,6)$. Then $r = (1,1,2,4,4,\mathbf{4},4,7,7,7,8,9)$.

We point out that the extra $\{k_1+1\}$ in the definition of the sequence r is needed to be able to reconstruct the original sequences s and t from a given r. Note that the length of r is therefore $(\lambda_1-1)+(\lambda_2-1)+1$.

Further, the product $\star_{p_1,\dots,p_i}$ is defined similarly, with the difference that the two words u and v are combined such that u is inserted into the positions $p_1,\dots,p_i$ of the new word. More precisely, $z = (u,s)\star_{p_1,\dots,p_i}(v,t)$, where

- $z_{p_i} = u_i$ and $z_{\ell_j} = v_j$, where $(\ell_1,\dots,\ell_j) = (1,2,\dots,n)\setminus(p_1,\dots,p_i)$ and
- $r = s \cup \{k_1+1\} \cup t^{\rightarrow}$.

Finally, we prove a recurrence relation for the special case $\mathcal{F}^1_{n,x}(\alpha,\beta,\gamma,a,b) = \sum_{k=0}^{n} x^k G^k_n(\alpha,\beta,\gamma)$.

Theorem 10. *For n, k, α, β, γ positive integers with $k \le n$, $\alpha|\beta$, $\alpha|\gamma$, and a, b, x indeterminate, we have*

$$\mathcal{F}^1_{n+1,x}(\alpha,\beta,\gamma,a,b) = \gamma\mathcal{F}^1_{n,x}(\alpha,\beta,\gamma-\alpha,a,b) + xb\beta\sum_{i=0}^{n}\binom{n}{i}\mathcal{F}^1_{i,x}(\alpha,\beta,\gamma,a,b)\mathcal{F}^1_{n-i,x}(\alpha,\beta,\beta,a,b).$$

Proof. Consider now (u,s)-pairs where s is empty. For convenience, we talk now simple about words, u, and not about the pairs (u,s). Let u_{n+1} denote the $(n+1)$th entry of a word u. This can be a symbol of $[\gamma]$ or a symbol from a β-group. In the first case, i.e., the weighted sum of all words with $u_{n+1} \in [\gamma]$ is given by $\gamma\mathcal{F}^1_{n,x}(\alpha,\beta,\gamma-\alpha,a,b)$, since there cannot be any symbol from the α-block of u_{n+1} but other way u_{n+1} does not influence the weight of the word.

Consider now a word, u, where u_{n+1} is from a β-group. There are β possibilities to choose a value for u_{n+1}. The remaining part of the word $u_1\cdots u_n$ can be obtained as a star-product of a word of length i over the alphabet $\mathcal{A}(k-1,\beta,\gamma)$ and a word of length $n-i$ over the alphabet $\mathcal{A}(k-1,\beta,\beta)$ where the kth β-group can occur only on the positions of the second word. However, the weight of such a star-product does not take into account this kth β group, hence we need to multiply the product of weights in this case by xb. □

4. Applications

We point now out some of the special cases that our results include.

4.1. Generalized Stirling numbers and related number sequences

First, we note that $G^k_n(\alpha,\beta,\gamma,a,b)$ is a generalization (a weighted version) of the expression $k!\beta^k S^k_n(\alpha,\beta,\gamma)$, where $S^k_n(\alpha,\beta,\gamma)$ denotes the generalized Stirling numbers introduced by Hsu and Shiue [15] for real or complex α,β,γ as the pair

$$(z|\alpha)_n = \sum_{k=0}^{n} S^k_n(\alpha,\beta,\gamma)(z-\gamma|\beta)_k,$$

$$(z|\beta)_n = \sum_{k=0}^{n} S^k_n(\beta,\alpha,-\gamma)(z+\gamma|\alpha)_k.$$

The following exponential generating function formula for the array $S^k_n(\alpha,\beta,\gamma)$ is known

$$\sum_{n=0}^{\infty} S^k_n(\alpha,\beta,\gamma)\frac{t^n}{n!} = (1+\alpha t)^{\frac{\gamma}{\alpha}}\frac{1}{k!}\left[\tfrac{(1+\alpha t)^{\frac{\beta}{\alpha}}-1}{\beta}\right]^k.$$

We define the weighted generalized Stirling numbers, $S^k_n(\alpha,\beta,\gamma,a,b)$, by the generating function

$$\sum_{n=0}^{\infty} S^k_n(\alpha,\beta,\gamma,a,b)\frac{t^n}{n!} = \left(\frac{(b(1+\alpha t)^{\frac{\beta}{\alpha}}-a)}{\beta}\right)^k\frac{(1+\alpha t)^{\frac{\gamma}{\alpha}}}{k!}.$$

Then,

$$G^k_n(\alpha,\beta,\gamma,a,b) = k!\beta^k S^k_n(\alpha,\beta,\gamma,a,b).$$

For instance, the identity (7) in [15] follows immediately from Theorem 4 by setting $a = b = 1$. The generalized Stirling numbers, $S_n^k(\alpha, \beta, \gamma)$, are well studied objects [2,3,7,17,27,28,34,36]. For instance, in [3] the authors present, the unfair distribution model" for these numbers and show also the connections to other known models. Since the generalized Stirling numbers itself generalize and unify numerous classes of numbers (see for instance the references in [3]), our results on the weighted enumeration imply several results on weighted versions of those special number sequences.

4.2. The non-degenerate case

Our model can be easily modified in order to give combinatorial proofs for the unified Apostol type Bernoulli, Euler, Genocchi, and Fubini polynomials defined by Kurt [24]. Formally, this case corresponds to $\alpha = 0$ and $\beta = 1$.

Let the alphabet be $\{1, 2, \dots, k, k+1, \dots, k+\gamma\}$ and each symbol of $[k]$ may arise arbitrary often while the symbols $\{k+1, k+2, \dots, k+\gamma\}$ at most once. Our previous proofs, respectively proof ideas, can be adapted to this case. We show one example of this adaptation.

Theorem 11. *For n, k, γ positive integers with $k, \gamma \leq n$, and a, b, x indeterminates, we have*

$$\mathcal{F}_{n,x}^{1}(0, 1, \gamma, a, b) = \sum_{\ell=0}^{n} \binom{n}{\ell} \sum_{k=0}^{\ell} (-1)^k x^k k! S_\ell^k(0, 1, 0, a, b) \gamma^{n-\ell}. \tag{6}$$

Proof. On a word of length n over the alphabet $[k+\gamma]$ say ℓ positions are from $[k]$ and the remaining $n-\ell$ positions are from $[\gamma]$. There are $\binom{n}{\ell}$ ways of choosing the ℓ positions. The weight of a word restricted on the ℓ positions with *at most* k different symbols is $x^k b^i (b-a)^{k-i}$, if i different symbols of the possible $[k]$ arise in the word particularly. For the remaining $n-\ell$ positions we can choose the symbols in $\gamma^{n-\ell}$ ways. Hence, a weighted allocation of the letters into the ℓ positions can be done in $x^k G_n^k(0, 1, 0, a, b)\gamma^{n-\ell}$ ways. Application of the inclusion–exclusion principle completes the proof. □

4.3. Higher order generalized geometric polynomials

Setting $a = 1$ and $b = 1$ in (3) we obtain higher-order generalized geometric polynomials defined by Kargin–Cekim in [16].

In [30] the authors present a combinatorial model for the polynomials in terms of barred preferential arrangements. A barred preferential arrangement is an ordered partition with bars inserted between the blocks. In the generalized model each block contains certain number (β or γ) of further compartments and there is a restriction on how the compartments can be filled, i.e., from α compartments only one can be occupied. Without recalling the details precisely of this model we mention here the equivalence of that model with α-distanced word presented here. Namely, if we allow for the k blocks with β compartments to be possible empty and associate the weight $b^i(b-a)^{k-i}$ to a barred preferential arrangement where i blocks with β compartments are non-empty we obtain an equivalent combinatorial interpretation of (3).

On the other hand, our results include identities of higher-order generalized geometric polynomials. For instance, when $(\alpha, \beta, \gamma, \lambda, a, b, x) = (\alpha, \beta, \gamma, \lambda, 1, 1, x)$ on (5) we obtain (3.1) of [16], and (16) of [30]. On Theorem 8 when $(\alpha, \beta, \gamma, \lambda, a, b, x) = (\alpha, \beta, \gamma, \lambda, 1, 1, x)$ we obtain Theorem 3.4 in [30].

5. Conclusion

In this work we introduced a generalization of the unified Apostol–Fubini polynomials of higher order and studied them using combinatorial methods. We defined α-distanced words over alphabets that fulfill some special conditions. We showed some properties of the sum of weights of all words of length n. Our approach can be extended, as for instance, considering a generalization of the ω-torsion Fubini polynomials studied in [19] is

$$\sum_{n=0}^{\infty} \mathcal{F}_{n,x}^{\lambda,\omega}(\alpha, \beta, \gamma, a, b) \frac{t^n}{n!} = \frac{(1+\alpha t)^{\frac{\gamma}{\alpha}}}{[1 - x^\omega [b(1+\alpha t)^{\frac{\beta}{\alpha}} - a]^\omega]^\lambda}. \tag{7}$$

The combinatorial treatment of these polynomials can be made by adapting our previous investigations.

Data availability

No data was used for the research described in the article.

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